

Existence and properties of solutions for neural field equations

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The first goal of this work is to study the solvability of the neural field equation (known as 'Amari equation')

$$\tau \frac{\partial u(\mathbf{x}, t)}{\partial t} + u(\mathbf{x}, t) = \int_{\mathbb{R}^m} w(\mathbf{x}, \mathbf{y}) f(u(\mathbf{y}, t)) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^m, \quad t > 0$$

which is an integro-differential equation in $m+1$ dimensions. In particular, we show the existence of global solutions for smooth activation functions f with values in $[0, 1]$ and L^1 kernels w via the Banach fixpoint theorem. We note that this setting is much more general than in most related studies, e.g. Ermentrout and McLeod (*Proceedings of the Royal Society of Edinburgh* 1993; 123A:461–478).

For a Heaviside-type activation function f , we show that the approach above fails. However, with slightly more regularity on the kernel function w (we use Hölder continuity with respect to the argument \mathbf{x}) we can employ compactness arguments, integral equation techniques and the results for smooth nonlinearity functions to obtain a global existence result in a weaker space.

Finally, general estimates on the speed and durability of waves are derived. We show that compactly supported waves with directed kernels (i.e. $w(\mathbf{x}, \mathbf{y}) \leq 0$ for $\mathbf{x} \leq \mathbf{y}$) decay exponentially after a finite time and that the field has a well-defined finite speed. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

Modeling neurodynamics has a long tradition in mathematical biology and computational neuroscience, starting with the study of simple neuron models and the theory of neural networks in the 1940s [1–8]. One particular neuron model with certain physiological significance is the *leaky integrator unit* [1–3, 5, 7, 8] described by the ODEs

$$\tau \frac{du_i(t)}{dt} + u_i(t) = \sum_{j=1}^N w_{ij} f(u_j(t)) \quad (1)$$

Here $u_i(t)$ denotes the time-dependent membrane potential of the i th neuron in a network of N units with synaptic weights w_{ij} . The nonlinear function f describes the conversion of the membrane potential $u_i(t)$ into a spike train $r_i(t) = f(u_i(t))$, and is called the *activation function*.

The left-hand side of Equation (1) describes the intrinsic dynamics of a leaky integrator unit, i.e. an exponential decay of membrane potential with time constant τ . The right-hand side of Equation (1) represents the *net-input* to unit i : the weighted sum of activity delivered by all units j that are connected to unit i ($j \rightarrow i$). Therefore, the weight matrix $W = (w_{ij})$ comprises three different kinds of information: (1) unit j is connected to unit i if $w_{ij} \neq 0$ (connectivity, network topology) and (2) the *synapse* $j \rightarrow i$ is *excitatory* ($w_{ij} > 0$), or *inhibitory* ($w_{ij} < 0$), (3) the *strength* of the synapse is given by $|w_{ij}|$.

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For the activation function f , essentially two different approaches are common. On the one hand, a *deterministic* McCulloch–Pitts neuron [6] is obtained from a Heaviside step function

$$f(s) := \begin{cases} 0, & s < \eta \\ 1, & s \geq \eta \end{cases} \quad (2)$$

for $s \in \mathbb{R}$ with an *activation threshold* η describing the *all-or-nothing-law* of action potential generation. Supplementing Equation (1) with a resetting mechanism for the membrane potential, the Heaviside activation function provides a *leaky integrate and fire* neuron model [1].

On the other hand, a *stochastic* neuron model leads to a continuous activation function $f(s) = \text{Prob}(s \geq \eta)$ describing the probability that a neuron fires if its membrane potential is above threshold [1]. In computational neuroscience this probability is usually approximated by the sigmoidal *logistic function*

$$f(s) = \frac{1}{1 + e^{-(s-\eta)}} \quad (3)$$

Analyzing and simulating large neural networks with complex topology are a very hard problem, due to the nonlinearity of f and the large number of synapses (approx. 10^4 per neuron) and neurons (approx. 10^{12}) in human cortex. Instead of analytically or numerically computing the sum in the right-hand side of Equation (1), substituting it by an integral over a continuous neural tissue, often facilitates such examinations. Therefore, continuum approximations of neural networks have been proposed since 1960s [1, 9–22].

Starting with the leaky integrator network equation (1), the sum over all units is replaced by an integral transformation of a neural field quantity $u(x, t)$, where the continuous parameter $x \in \mathbb{R}^m$ now indicates the position i in the network. Correspondingly, the synaptic weight matrix w_{ij} turns into a kernel function $w(x, y)$. Then, Equation (1) assumes the form of a *neural field equation (NFE)* as discussed in [9, 21]

$$\tau \frac{\partial u(x, t)}{\partial t} + u(x, t) = \int_{\mathbb{R}^m} w(x, y) f(u(y, t)) dy, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (4)$$

with *initial condition*

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^m \quad (5)$$

Therefore, Equation (4) is often called the *Amari equation* in the literature.

Up to now, NFEs have been mainly investigated under serious restrictions upon the integral kernel $w(x, y)$, namely homogeneity ($w(x, y) = w(x - y)$) and isotropy ($w(x, y) = w(|x - y|)$). In these cases, the technique of Green's functions allows the derivation of PDEs for the neural waves $u(x, t)$ assuming special kernels such as exponential, locally uniform or 'Mexican hat' functions [12, 15, 16, 19, 20]. Solutions for such NFEs have been obtained for traveling waves [12, 13], and for macroscopic, stationary neurodynamics in order to predict spectra of the electroencephalogram [18, 19, 22], or bimanual movement coordination patterns [16].

By contrast, heterogeneous kernels and thalamo-cortical loops in addition to homogeneous cortico-cortical connections have been discussed in [10, 11, 22–26]. However, at present there is no universal neural field theory available, that would allow the study of field equations with general synaptic kernel functions. Yet such a theory would be mandatory for modeling mesoscopic and transient neurodynamics as is characteristic, e.g. for cognitive phenomena.

Our goal here is to provide general analytical results for the inhomogeneous Amari field equation, significantly extending the works [12, 13] that consider only special kernels. We examine the solvability of the *integro-differential equation* (4) with tools from functional analysis, the theory of ordinary differential equations (ODEs) and integral equations. We provide a proof of global existence of solutions and study their properties in dependence on the smoothness of the synaptic kernel function w and the smoothness of the activation function f . In particular, we provide a detailed discussion of the case when the activation function has a jump, where the standard tools are no longer applicable.

We expect that our analysis will serve as a model for various variations and generalizations of NFEs (in particular, when propagation times are included), which are currently being investigated for applications in the field of cognitive neurodynamics [27].

2. The neural field equation

For studying the existence of solutions of the NFE (4), we define the operator

$$(Fu)(x, t) := \frac{1}{\tau} \left(-u(x, t) + \int_{\mathbb{R}^m} w(x, y) f(u(y, t)) dy \right), \quad x \in \mathbb{R}^m, \quad t > 0 \quad (6)$$

Then the NFE (4) can be reformulated as

$$u' = Fu \quad (7)$$

where u' denotes the derivative of u with respect to the time variable t . For later use we also define the operators

$$(Au)(x, t) := \int_0^t (Fu)(x, s) ds, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (8)$$

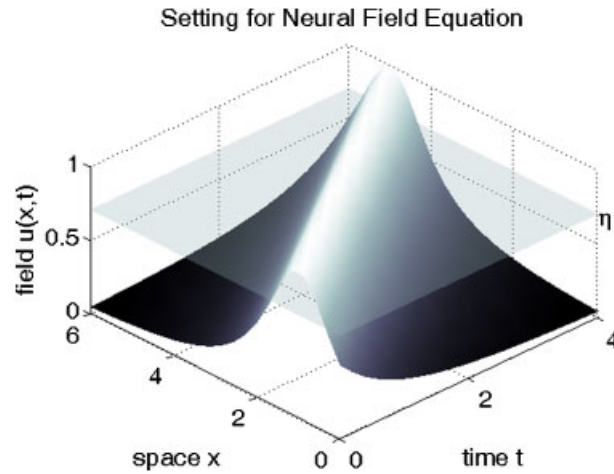


Figure 1. Setting for the neural field equation (4) for the case $m=1$. The potential $u(x,t)$ is depending on space $x \in \mathbb{R}^m$ and time $t \geq 0$. Here, a pulse is traveling in the x -direction when time increases. The plane indicates the cut-off parameter η in the activation function f . Only a field $u(x,t) \geq \eta$ will contribute to the increase of the potential.

and

$$(Ju)(x,t) := \frac{1}{\tau} \int_{\mathbb{R}^m} w(x,y)f(u(y,t)) dy, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (9)$$

To define appropriate spaces and study the mapping properties of the operators F and A we need to formulate conditions on the synaptic weight kernel w and the activation function f in the NFE. Here, we will study two classes of functions f .

The *first* class contains smooth functions f . In this case we can employ tools from the classical theory of ODEs to obtain existence results.

The *second* class works with non-smooth functions f , as, for example, when f is a Heaviside jump function. In this case the above theory is not applicable and we will construct counterexamples. We will study the existence problem by investigating particular kernels w that allow particular solutions (Figure 1).

2.1. General estimates for solutions to the NFE

The goal of this section is to derive general estimates for solutions to the NFE. We will see that the solutions are nicely bounded under fairly general conditions on the kernel and activation function and that we can estimate the speed and behavior of compactly supported pulses.

The following conditions on the kernel w and the activation function f will be the general conditions used throughout this work.

Definition 2.1 (Synaptic kernel and activation function)

Let the synaptic integral kernel w satisfy

$$w(x, \cdot) \in L^1(\mathbb{R}^m) \quad \forall x \in \mathbb{R}^m \quad (10)$$

$$\sup_{x \in \mathbb{R}^m} \|w(x, \cdot)\|_{L^1(\mathbb{R}^m)} \leq C_w \quad (11)$$

$$\|w(x, \cdot) - w(\tilde{x}, \cdot)\|_{L^1(\mathbb{R}^m)} \leq c_w |x - \tilde{x}|, \quad x, \tilde{x} \in \mathbb{R}^m \quad (12)$$

with some constant $C_w > 0$ and

$$|w(x,y)| \leq C_\infty, \quad x,y \in \mathbb{R}^m \quad (13)$$

For the function $f: \mathbb{R} \rightarrow \mathbb{R}$ we assume that

$$f(\mathbb{R}) \subset [0, 1] \quad (14)$$

We use the definition $BC^{0,\alpha}(\mathbb{R}^m, L^1(\mathbb{R}^m))$ for $\alpha \in (0, 1]$ for the set of functions $w(x,y)$ for which (10)–(12) with $|x - \tilde{x}|$ replaced by $|x - \tilde{x}|^\alpha$ are satisfied. Then, we can equip this space with the norm

$$\|w\|_{BC^{0,\alpha}(\mathbb{R}^m, L^1(\mathbb{R}^m))} := \sup_{x \in \mathbb{R}^m} \|w(x, \cdot)\|_{L^1(\mathbb{R}^m)} + \sup_{x \neq \tilde{x}} \int_{\mathbb{R}^m} \frac{|w(x,y) - w(\tilde{x},y)|}{|x - \tilde{x}|^\alpha} dy \quad (15)$$

Further, for some results we assume that the kernel w is sensitive to any open set $G \subset \mathbb{R}^m$ in the sense that

$$\sup_{x \in \mathbb{R}^m} \left| \int_G w(x, y) dy \right| > 0 \quad (16)$$

The NFE allows general and global estimates for the above kernels, which also guarantee existence of solutions.

Lemma 2.2

Let $u_0 \in BC(\mathbb{R}^m)$ be an initial field (5) for the NFE (4) and assume that the kernel w satisfies the conditions of Definition 2.1. Then the solution $u(x, t)$ to the NFE is bounded by

$$C_{\text{tot}} := \max(\|u_0\|_\infty, |C_w|) \quad (17)$$

i.e. we have the *general estimate*

$$|u(x, t)| \leq C_{\text{tot}}, \quad x \in \mathbb{R}^m, \quad t \geq 0 \quad (18)$$

for solutions to the NFE.

Proof

We first note that the term Ju defined in (9) can be estimated by

$$|(Ju)(x, t)| \leq \frac{C_w}{\tau} \quad (19)$$

Next, we observe that the derivative $u'(t)$ in the NFE is bounded by

$$u'(x, t) \leq -bu(x, t) + c, \quad u'(x, t) \geq -bu - c \quad (20)$$

with $b = 1/\tau$ and $c = C_w/\tau$. Thus, the value of $u(x, \cdot)$ for x fixed will be bounded by the solution to the ODE (A1) with $a = u_0(x)$, $b = 1/\tau$ and $c = C_w/\tau$. According to Lemma 4.1 the bound is given by C_{tot} defined in (17). This proves the estimate (18). \square

2.2. The NFE with a smooth activation function f

Here, for the function $f: \mathbb{R} \rightarrow \mathbb{R}$ we assume that

$$f \in BC^1(\mathbb{R}) \quad (21)$$

With the conditions (10)–(14) we now obtain the following mapping properties of the neural field operator F . Here, we define the anisotropic space $BC(\mathbb{R}^m, C^1(\mathbb{R}_0^+))$ to be the linear function space of all functions $f(x, t)$ which are bounded and continuous in x and continuously differentiable in t with bounded derivative equipped with the norm

$$\|f\|_{BC(\mathbb{R}^m, C^1(\mathbb{R}_0^+))} := \sup_{x \in \mathbb{R}^m, t \in \mathbb{R}_0^+} \left(|f(x, t)| + \left| \frac{df}{dt}(x, t) \right| \right) \quad (22)$$

Lemma 2.3

The operator F defined by (6) with kernel w and activation function f which satisfy the conditions of Definition 2.1 and (21) is a bounded nonlinear operator on $BC(\mathbb{R}^m, C^1(\mathbb{R}_0^+))$, i.e. it maps bounded sets into bounded sets.

Proof

To prove boundedness we need to estimate the integral operator. The term $f(u(y, t))$ has values in $[0, 1]$, thus we can estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^m} w(x, y) f(u(y, t)) dy \right| &\leq \int_{\mathbb{R}^m} |w(x, y)| dy \\ &\leq \|w(x, \cdot)\|_{L^1(\mathbb{R}^m)} \\ &\leq C_w \end{aligned} \quad (23)$$

for all $x \in \mathbb{R}^m$. This proves that for u bounded the function $Fu \in L^\infty(\mathbb{R}^m)$. The continuity of $(Fu)(x, t)$ with respect to x and the differentiability with respect to t is obtained as follows. We use (12) to estimate

$$\begin{aligned} |(Ju)(x, t) - (Ju)(\tilde{x}, t)| &\leq \frac{1}{\tau} \int |w(x, y) - w(\tilde{x}, y)| |f(u(y, t))| dy \\ &\leq \frac{1}{\tau} \int |w(x, y) - w(\tilde{x}, y)| dy \\ &\leq \frac{C_w}{\tau} |x - \tilde{x}| \end{aligned}$$

Since $u(x, t)$ is continuous in x we obtain the continuity of Fu in x . Finally, we need to show that Fu is continuously differentiable with respect to the time variable. This is clear for the first term $-u(x, t)/\tau$. The time dependence of the integral

$$(Ju)(x, t) := \int_{\mathbb{R}^m} w(x, y) f(u(y, t)) dy \quad (24)$$

is implicitly given by the time dependence of the field $u(y, t)$. By assumption we know that $u(x, \cdot) \in C^1(\mathbb{R}^+)$ and the function f is $BC^1(\mathbb{R})$. Then via the *chain rule* we derive

$$\frac{d}{dt} f(u(y, t)) = \left. \frac{df(s)}{ds} \right|_{s=u(y, t)} \cdot \frac{\partial u(y, t)}{\partial t}$$

Since f' is bounded on \mathbb{R} and w is integrable we obtain the differentiability of the integral with the derivative

$$\frac{\partial Ju}{\partial t}(x, t) = \int_{\mathbb{R}^m} w(x, y) \left\{ \frac{df}{ds}(u(y, t)) \cdot \frac{\partial u}{\partial t}(y, t) \right\} dy, \quad t > 0 \quad (25)$$

The function $dJu/dt(x, t)$ depends continuously on $t \in \mathbb{R}^+$ due to the continuity of df/ds and du/dt in t and the term (25) is bounded for $t \geq 0$ and $x \in \mathbb{R}^m$. This completes the proof. \square

By integration with respect to t we equivalently transform the NFE (4) or (7), respectively, into a *Volterra integral equation*

$$u(x, t) = u(x, 0) + \int_{s=0}^t (Fu)(x, s) ds, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (26)$$

which, with A defined in (8), can be written in the form

$$u(x, t) = u(x, 0) + (Au)(x, t), \quad x \in \mathbb{R}^m, \quad t > 0 \quad (27)$$

Lemma 2.4

The Volterra equation (26) or (27), respectively, is solvable on $\mathbb{R}^m \times (0, \rho)$ for some $\rho > 0$ if and only if the NFE (4) or (7), respectively, is solvable for $x \in \mathbb{R}^m$ and $t \in (0, \rho)$. In particular, solutions to the Volterra equation (26) are in $BC^1(\mathbb{R}_0^+)$.

Proof

If the NFE is solvable with some continuous function $u(x, t)$, we obtain the Volterra integral equation (26) for the solution u by integration.

To show that a solution $u(x, t)$ to the Volterra integral equation (26) in $BC(\mathbb{R}^m \times \mathbb{R}_0^+)$ satisfies the NFE (4) we first need to ensure sufficient regularity, since solutions to Equation (4) need to be differentiable with respect to t . We note that the function

$$g_x(t) := \int_0^t (Fu)(x, s) ds, \quad t > 0$$

is differentiable with respect to t with continuous derivative for each $x \in \mathbb{R}^m$. Thus, the solution $u(x, t)$ to Equation (26) is continuously differentiable with respect to $t > 0$ and the derivative is continuous on $[0, \infty)$. Now, the derivation of (4) for u from (26) is straightforward by differentiation. \square

An important preparation for our local existence study is the following lemma. We need an appropriate local space, which for $\rho > 0$ is chosen as

$$X_\rho := BC(\mathbb{R}^m \times [0, \rho]) \quad (28)$$

understood as the space of functions $u(x, t)$ with $x \in \mathbb{R}^m$ and $t \in [0, \rho]$ equipped with the norm

$$\|u\|_\rho := \sup_{x \in \mathbb{R}^m, t \in [0, \rho]} |u(x, t)| \quad (29)$$

is a Banach space. For $\rho = \infty$ we denote this space by X , i.e.

$$X := BC(\mathbb{R}^m \times \mathbb{R}_0^+), \quad \|u\|_X := \sup_{x \in \mathbb{R}^m, t \in \mathbb{R}_0^+} |u(x, t)| \quad (30)$$

An operator \tilde{A} from a normed space X into itself is called a *contraction*, if there is a constant q with $0 < q < 1$ such that

$$\|\tilde{A}u_1 - \tilde{A}u_2\| \leq q \|u_1 - u_2\| \quad (31)$$

is satisfied for all $u_1, u_2 \in X$. A point $u_* \in X$ is called *fixed point* of \tilde{A} if

$$u_* = \tilde{A}u_* \quad (32)$$

is satisfied. We are now prepared to study the properties of A on X_ρ .

Lemma 2.5

For $\rho > 0$ chosen sufficiently small, the operator A is a contraction on the space X_ρ defined in (28).

Proof

We estimate $Au_1 - Au_2$ and abbreviate $u := u_1 - u_2$. We decompose $A = A_1 + A_2$ into two parts with the linear operator

$$(A_1 v)(x, t) := \frac{-1}{\tau} \int_0^t v(x, s) ds, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (33)$$

and the nonlinear operator

$$(A_2 v)(x, t) := \frac{1}{\tau} \int_0^t \int_{\mathbb{R}^m} w(x, y) f(v(y, s)) dy ds, \quad x \in \mathbb{R}^m, \quad t > 0 \quad (34)$$

We can estimate the norm of A_1 by

$$\|A_1 u\|_\rho \leq \frac{\rho}{\tau} \|u\|_\rho \quad (35)$$

which is a contraction if ρ is sufficiently small. Since $f \in BC^1(\mathbb{R})$ there is a constant L such that

$$|f(s) - f(\tilde{s})| \leq L|s - \tilde{s}|, \quad s, \tilde{s} \in \mathbb{R} \quad (36)$$

This yields

$$\begin{aligned} |Ju_1(x, t) - Ju_2(x, t)| &\leq \frac{1}{\tau} \int_{\mathbb{R}^m} |w(x, y)| |f(u_1(y, t)) - f(u_2(y, t))| dy \\ &\leq \frac{1}{\tau} L \int_{\mathbb{R}^m} |w(x, y)| |u_1(y, t) - u_2(y, t)| dy \\ &\leq \frac{1}{\tau} LC_w \|u_1 - u_2\|_\infty \end{aligned} \quad (37)$$

Finally, by an integration with respect to t we now obtain the estimate

$$\|A_2 u_1 - A_2 u_2\|_\rho \leq \frac{\rho}{\tau} LC_w \|u_1 - u_2\|_\infty \quad (38)$$

For ρ sufficiently small the operator A_2 is a contraction on the space X_ρ . For

$$q := \frac{\rho}{\tau} (1 + LC_w) < 1 \quad (39)$$

the operator $A = A_1 + A_2$ is a contraction on X_ρ . □

Now, the local existence theorem is given by the following theorem.

Theorem 2.6 (Local existence for NFE)

Assume that the synaptic weight kernel w and the activation function f satisfy the conditions of Definition 2.1 and (21) and let $\rho > 0$ be chosen such that (39) is satisfied with L being the Lipschitz constant of f . Then we obtain existence of solutions to the NFEs on the interval $[0, \rho]$.

Remark

The result is a type of Picard–Lindelöf theorem for the NFE (4) under the conditions of Definition 2.1 and (21).

Proof

We employ the Banach Fix-Point Theorem to the operator equation (27). We have shown that the operator A is a contraction on X_ρ defined in (28). Then, also the operator $\tilde{A}u := u_0 + Au$ is a contraction on the complete normed space X_ρ . Now, according to the Banach fixpoint theorem the equation

$$u = \tilde{A}u \quad (40)$$

as a short form of the Volterra equations (27) or (26), respectively, has one and only one fixpoint u_* . This proves the unique solvability of (26). Finally, by the equivalence Lemma 2.4 we obtain the unique solvability of the NFE (4) on $t \in [0, \rho]$. □

In a last part of this section we combine the global estimates with local existence to obtain a global existence result.

Theorem 2.7 (Global existence of solutions to NFE)

Under the conditions of Definition 2.1 we obtain existence of global bounded solutions to the NFE.

Proof

We first remark that the NFE does not explicitly depend on time. As a result we can apply the local existence result with the same constant ρ to any interval $[t_0, t_0 + \rho] \subset \mathbb{R}$ when initial conditions $u(x, t_0) = \tilde{u}_0$ for $t = t_0$ are given. This means that we can use Theorem 2.6 iteratively.

First, we obtain existence of a solution on an interval $I_0 := [0, \rho]$ for

$$\rho := \frac{\tau}{2(1 + LC_w)} \tag{41}$$

Then, the function $u_1(x) := u(x, \rho)$ serves as new initial condition for the NFE on $t > \rho$ with initial conditions u_1 at $t = \rho$. We again apply Theorem 2.6 to this equation to obtain the existence of a solution on the interval $I_1 = [\rho, 2\rho]$.

This process is continued to obtain existence on the intervals $I_n := [n\rho, (n+1)\rho]$, $n \in \mathbb{N}$, which shows existence for all $t \in \mathbb{R}$. Global bound for this solution have been derived in Lemma 2.2. \square

2.3. The NFE with a heaviside activation function f

In this section we will construct special solutions to the NFE in the case of an activation function f given by Equation (2). In this case the results of the preceding sections are no longer applicable. We will develop specific methods to analyze the solvability of the equation for this particular case.

We first show that for the activation function f defined in (2), the operator F no longer depends continuously on the function u .

Lemma 2.8

With f given by (2), w according to Definition 2.1 and the additional condition (16) for the kernel, the function Fu does not depend continuously on $u \in X$ with X defined in (30).

Proof

Consider the sequence $(u_n)_{n \in \mathbb{N}}$ of functions $u_n \in X$ with

$$u_n(x, t) := \begin{cases} 0, & x \leq -2 \\ \left(\eta - \frac{1}{n}\right) \cdot (2+x), & x \in (-2, -1) \\ \eta - \frac{1}{n}, & x \in [-1, 1] \\ \left(\eta - \frac{1}{n}\right) \cdot (2-x), & x \in (1, 2) \\ 0, & x \geq 2 \end{cases} \tag{42}$$

for $x \in \mathbb{R}$ and $t \geq 0$, compare Figure 2. The function u is defined by (42) with $n = \infty$, where we use $1/\infty = 0$. Then $u_n \rightarrow u$ for $n \rightarrow \infty$ in X . For all $n \in \mathbb{N}$ we have $Fu_n = -u_n/\tau$, since $f(u_n(y, t)) = 0$ for $y \in \mathbb{R}^m$ and $t \geq 0$. However, we calculate

$$(Fu)(x, t) = \frac{-1}{\tau} u(x, t) + \underbrace{\frac{1}{\tau} \int_{[-1,1]} w(x, y) dy}_{=J\eta(x)} \tag{43}$$

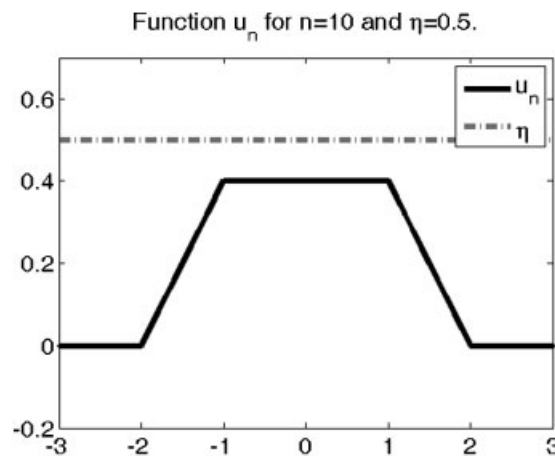


Figure 2. Function u_n used to prove the non-continuity of the operator F for a Heaviside-type activation function f in the Amari equation.

Thus, we have

$$\lim_{n \rightarrow \infty} (Fu_n(x, t) - Fu(x, t)) = J\eta(x), \quad x \in \mathbb{R} \quad (44)$$

i.e. for general kernels $w(x, y)$ where $J\eta(x) \neq 0$ the operator F is not continuous. \square

Remark

As a consequence of Lemma 2.8 the operator A is not a contraction on X_ρ for any $\rho > 0$, since

$$\begin{aligned} |Au_n(t) - Au(t)| &= \left| \frac{-1}{\tau} \int_0^t (u_n(x, s) - u(x, s)) ds + \int_0^t \int_{\mathbb{R}} w(x, y) (f(u_n(y, s)) - f(u(y, s))) dy ds \right| \\ &\rightarrow |J(x)| t, \quad n \rightarrow \infty \end{aligned} \quad (45)$$

where $J(x)$ is given by (43).

Since the operator A does not depend Lipschitz continuously on u , we need to use techniques different from the Banach fixpoint theorem above. Here, we will develop an approach based on compactness arguments to carry over the existence results from above to the non-smooth Heaviside activation function f . To this end we define the Hölder space

$$X_{\rho, \alpha} := BC^\alpha(\mathbb{R}^m \times [0, \rho]) \quad (46)$$

for $\alpha \in (0, 1]$ understood as the set of functions $u(x, t)$ defined on $\mathbb{R}^m \times [0, \rho]$ equipped with the Hölder norm

$$\|\varphi\|_{\rho, \alpha} := \|\varphi\|_\rho + \sup_{t \in [0, \rho], x, y \in \mathbb{R}^m} \frac{|\varphi(x, t) - \varphi(y, t)|}{|x - y|^\alpha} + \sup_{x \in \mathbb{R}^m, t, s \in [0, \rho]} \frac{|\varphi(x, t) - \varphi(x, s)|}{|t - s|^\alpha} \quad (47)$$

It is well known that the Hölder space on a compact set M is compactly embedded into the space $BC(M)$. However, for unbounded sets like the space \mathbb{R}^m this is not the case. However, we still get *local compactness* of the embedding, i.e. every bounded sequence $(\psi_n)_{n \in \mathbb{N}}$ in $X_{\rho, \alpha}$ does have a subsequence $(\tilde{\psi}_k)_{k \in \mathbb{N}}$ which is *locally converging* in X_ρ toward an element $\psi \in X_\rho$, i.e. where

$$\sup_{t \in [0, \rho], x \in B_R(0)} |\tilde{\psi}_k(x, t) - \psi(x, t)| \rightarrow 0, \quad n \rightarrow \infty \quad (48)$$

for every fixed $R > 0$. We need some of the mapping properties of the operators A_1 and A_2 defined in (33) and (34), respectively, in these spaces. This is the purpose of the following lemma. Define the *indicator function* of a set M by

$$\chi_M(x) := \begin{cases} 1, & x \in M \\ 0, & x \notin M \end{cases} \quad (49)$$

Lemma 2.9

The operator A_1 is a linear operator that maps X_ρ boundedly into X_ρ with norm bounded by ρ/τ . In particular, for $\rho < \tau$ the operator $I - A_1$ is invertible on X_ρ with bounded inverse given by

$$(I - A_1)^{-1} = \sum_{l=0}^{\infty} A_1^l \quad (50)$$

Moreover, the operators A_1 , $I - A_1$ and $(I - A_1)^{-1}$ are *local* with respect to the variable x with local bounds in the sense that

$$A_1(\chi_M u) = (\chi_M \cdot A_1)(u), \quad u \in X_\rho \quad (51)$$

for all open sets $M \subset \mathbb{R}^m$ where $\chi_M \cdot A_1$ is bounded in $BC(M \times [0, \rho])$ by ρ/τ . These operators map a locally convergent sequence onto a locally convergent sequence.

Proof

The linearity of A_1 is trivial and the bound of the operator A_1 has been derived in (35). Then the form (50) is the classical Neumann series in normed spaces. Clearly, the operator A_1 and $I - A_1$ are local in x in the sense of (51). And the bound ρ/τ holds for $\chi_M \cdot A_1$.

Consider a bounded locally convergent sequence $(\psi_n)_{n \in \mathbb{N}} \subset X_\rho$. Then we have

$$|A_1(\psi_n - \psi)(x, t)| \leq \frac{-1}{\tau} \int_0^t |\psi_n(x, s) - \psi(x, s)| ds \rightarrow 0, \quad n \rightarrow \infty \quad (52)$$

uniformly for $x \in B_R(0)$ and $t \in [0, \rho]$ for each fixed $R > 0$. This means that $A_1 \psi_n$ is a locally convergent sequence. The same arguments apply to $I - A_1$ and $(I - A_1)^{-1}$, and the proof is complete. \square

We have seen above that the operator F is not continuous on X or X_ρ , respectively. The same is true for the operator A_2 . However, we will see that the operators are bounded in appropriate spaces. Recall that for linear operators by basic functional analysis an operator is continuous if and only if it is bounded, for nonlinear operators boundedness and continuity are different.

Theorem 2.10

Let the kernel $w(x, y)$ be in $BC^{0,\alpha}(\mathbb{R}^m, L^1(\mathbb{R}^m))$, i.e. the function is Hölder continuous with respect to the first variable and integrable with respect to the second. Then the operator A_2 defined by (34) is a bounded operator from X_ρ into $X_{\rho,\alpha}$ defined in (46).

Proof

By direct estimates of the kernel of A_2 we obtain the boundedness of the operator on X_ρ . We basically have to estimate the Hölder norms of A_2u with respect to the space variable x and the time variable t for some function $u \in X_\rho$. First, we derive

$$\begin{aligned} |(A_2u)(x, t) - (A_2u)(x, s)| &= \left| \int_s^t \int_{\mathbb{R}^m} w(x, y) f(u(x, s)) dy ds \right| \\ &\leq |t - s| \int_{\mathbb{R}^m} |w(x, y)| dy \\ &\leq C_w |t - s| \end{aligned} \tag{53}$$

for $x \in \mathbb{R}^m$ and $t, s \in [0, \rho]$ with C_w given in Definition 2.1. Thus, the function A_2u is Lipschitz continuous with respect to t and by compact embedding of $BC^{0,1}([0, \rho])$ into $BC^{0,\alpha}([0, \rho])$ in every Hölder space for $\alpha \in (0, 1)$.

Hölder continuity for the x variable follows from the estimate

$$\begin{aligned} |(A_2u)(x, t) - (A_2u)(\tilde{x}, t)| &\leq \int_0^t \int_{\mathbb{R}^m} |w(x, y) - w(\tilde{x}, y)| dy ds \\ &\leq c\rho |x - \tilde{x}|^\alpha \end{aligned} \tag{54}$$

with some constant c according to our assumption on $w(x, \cdot)$. This completes the proof. □

We consider a sequence of nonlinear smooth functions $f_n: \mathbb{R} \rightarrow [0, 1]$ such that

$$f_n(t) = 0 \quad \text{on} \quad \left[-\infty, \eta - \frac{1}{n}\right], \quad f_n(t) = 1 \quad \text{on} \quad [\eta, \infty) \tag{55}$$

Such a sequence can be easily constructed with arbitrary degree of smoothness. We will denote the operators depending on the nonlinearity functions f_n by A_n and F_n and the operators with the function f by A and F , respectively. We split the operator A_n into $A_n = A_1 + A_{2,n}$. The operator A_2 with the discontinuity in the nonlinearity f generates some difficulties, which are reflected by the following result.

Lemma 2.11

For fixed $u \in X_\rho$ we have $A_{2,n}u \rightarrow A_2u$ locally. The convergence does not hold in the operator norm.

Proof

We estimate

$$\begin{aligned} \sigma_n &:= |A_2u(x, t) - A_{2,n}u(x, t)| \\ &\leq \left| \int_0^t \int_{\mathbb{R}^m} w(x, y) (f(u(y, t)) - f_n(u(y, t))) dy ds \right| \\ &\leq \int_0^t \int_{\mathbb{R}^m} |w(x, y)| |f(u(y, t)) - f_n(u(y, t))| dy ds \end{aligned}$$

Now with $M_n(t) := \{y \in \mathbb{R}^m : u(y, t) \in \text{supp}(f - f_n)\}$ we estimate this by

$$\sigma_n \leq \int_0^t \int_{M_n(t)} |w(x, y)| dy ds \rightarrow 0, \quad n \rightarrow \infty \tag{56}$$

as a result of (55). This holds uniformly on compact sets, but in general it does not hold uniformly for $x \in \mathbb{R}^m$. □

For some function $v \in X_\rho$ we define the set

$$M_{\eta, \rho, R}[v] := \{(y, s) \in \overline{B_R(0)} \times [0, \rho] : v(y, s) = \eta\} \tag{57}$$

i.e. $M_{\eta, \rho, R}[v]$ is the set of space-time points (y, s) in $B_R(0) \times [0, \rho]$ where $v(y, s)$ equals the threshold η in the Heaviside nonlinearity. When we use $R = \infty$ then in this definition $B_\infty(0)$ is equal to \mathbb{R}^m . By $\mu(M)$ we denote the Euclidean area, volume or more general *Euclidean measure*

$$\mu(M) := \int_M 1 dy \tag{58}$$

of a set M . We call an operator A_2 locally continuous if for a locally convergent sequence $u_n \rightarrow u$ we have $A_2u_n \rightarrow A_2u$.

Lemma 2.12

The operator A_2 is locally continuous in $v \in X_\rho$ if and only if the volume of $M_{\eta,\rho,\infty}[v]$ is zero. Moreover, in this case we have

$$u_n \xrightarrow{\text{loc}} u \Rightarrow A_{2,n}u_n \xrightarrow{\text{loc}} A_{2,n}u \tag{59}$$

Proof

We need to start with some preparations. We first note that when $\mu(M_{\eta,\rho,\infty}[v])$ is zero this is the case also for all $M_{\eta,\rho,R}[v]$ with $R > 0$. The set $M_{\eta,\rho,R}[v]$ is a closed set, thus $B_R(0) \setminus M_{\eta,\rho,R}[v]$ is an open set. We choose a sequence $G_l, l \in \mathbb{N}$ of closed sets $G_l \subset B_R(0) \setminus M_{\eta,\rho,R}[v]$ such that

$$\mu_l := \mu(B_R(0) \setminus G_l) \rightarrow 0, \quad l \rightarrow \infty$$

Second, if $v_n \rightarrow v$ locally in X_ρ , then for each $l \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $f(v_n(y,s)) = f(v(y,s)), (y,s) \in G_l$, for all $n \geq N$.

We are now prepared to prove continuity of A_2 in v . Let v be given with $\mu(M_{\eta,\rho,\infty}[v]) = 0$ and $(v_n)_{n \in \mathbb{N}}$ be a sequence in X_ρ with $v_n \rightarrow v$ locally. Given some $r > 0$ and $\varepsilon > 0$ we proceed as follows.

(1) We choose $R > 0$ such that

$$\rho \int_{\mathbb{R}^m \setminus B_R(0)} |w(x,y)| dy \leq \frac{\varepsilon}{2}, \quad x \in B_r(0)$$

The existence of such R is a consequence of the condition $w(x, \cdot) \in L^1(\mathbb{R}^m)$ which is continuous in $x \in \mathbb{R}^m$ and bounded on the compact set $\overline{B_r(0)}$.

(2) On $B_R(0)$ we choose $L \in \mathbb{N}$ such that

$$\mu_L \cdot C_\infty \leq \frac{\varepsilon}{2}$$

(3) Given L we choose N sufficiently large such that on G_L we have

$$f(v_n(y,s)) = f(v(y,s)), \quad (y,s) \in G_L \tag{60}$$

for all $n \geq N$.

We now estimate the integral

$$|A_2 v_n(x,t) - A_2 v(x,t)| \leq \left| \int_0^t \int_{\mathbb{R}^m} w(x,y)(f(v_n(y,s)) - f(v(y,s))) dy ds \right| \tag{61}$$

by a decomposition of the integration over \mathbb{R}^m into one over

$$M_1 := \mathbb{R}^m \setminus B_R(0), \quad M_2 := B_R(0) \setminus G_L, \quad M_3 := G_L$$

The three integrals can be estimated by (1)–(3) and we obtain

$$|A_2 v_n(x,t) - A_2 v(x,t)| \leq \varepsilon, \quad x \in B_r(0), \quad t \in [0, \rho], \quad n \geq N(\varepsilon) \tag{62}$$

This shows local continuity of A_2 in v .

If the volume of $M_\eta(v)$ is not zero, there is a set $G \subset \mathbb{R}^m \times [0, \rho]$ with $\mu(G) > 0$, where $v(y,s) = \eta$. In this case as in Lemma 2.8 we can construct a sequence of functions $v_n \in X_\rho$ which converges to v such that they are equal to v on $\mathbb{R}^m \times [0, \rho] \setminus G$ and $v_n(y,s) < \eta$ on the open interior of G . In this case we obtain a remainder term

$$|A_2 v_n(x,t) - A_2 v(x,t)| \rightarrow \left| \int_G w(x,y) dy ds \right| > 0, \quad n \rightarrow \infty$$

according to (16). This proves that in this case the operator A_2 is not continuous in v . The more general convergence (59) is shown with the same arguments, where the equality (60) needs to be replaced by some estimate involving f_n . \square

We will now carry out the basic steps to study solvability of the discontinuous equation.

We consider solutions $u_n \in X_\rho$ for some ρ with $\rho/\tau < 1$ of the Volterra equation (26) with function f_n for $n \in \mathbb{N}$, i.e.

$$u_n - Au_n = u_0, \quad n \in \mathbb{N} \tag{63}$$

Then, the operator $I - A_1$ is linear and invertible in X_ρ . Multiplication by the operator $(I - A_1)^{-1}$ leads to the equivalent equation

$$u_n - (I - A_1)^{-1} A_{2,n} u_n = (I - A_1)^{-1} u_0, \quad n \in \mathbb{N} \tag{64}$$

According to Lemma 2.2, the sequence $(u_n)_{n \in \mathbb{N}}$ of (4) on $[0, \rho]$ is bounded uniformly by the constant C_{tot} in X_ρ . Then, the sequence

$$\psi_n := A_{2,n} u_n, \quad n \in \mathbb{N} \tag{65}$$

is bounded in $X_{\rho,\alpha}$ for $\alpha > 0$. By the locally compact embedding of $X_{\rho,\alpha}$ into X_ρ , the sequence $(\psi_n)_{n \in \mathbb{N}}$ has a locally convergent subsequence in X_ρ which we denote by $(\psi_k)_{k \in \mathbb{N}}$ and its limit in X_ρ by ψ_* . The operator $(I - A_1)^{-1}$ maps locally convergent sequences onto locally convergent sequences, thus the sequence

$$u_k = (I - A_1)^{-1} u_0 + (I - A_1)^{-1} A_{2,k} u_k, \quad k \in \mathbb{N}$$

is locally convergent toward some function u_* . In this case by application of $I - A_1$ we obtain

$$u_* + A_1 u_* - \psi_* = u_0$$

If we could show that $A_2 u_* = \psi_*$, then we would obtain solvability of the equation $(I - A)u = u_0$ in X_ρ . However, in general we have

$$A_{2,k} u_k \not\rightarrow A_2 u_*, \quad k \rightarrow \infty$$

However, if $\mu(M_{\eta,\rho,\infty}(u_*)) = 0$ following Lemma 2.12 we obtain

$$A_{2,k} u_k \rightarrow A_2 u_*, \quad k \rightarrow \infty$$

therefore $\psi_* = A_2 u_*$. We summarize these results in the following theorem.

Theorem 2.13 (Local existence for Heaviside-type activation function f)

Consider a kernel w that satisfies the conditions of Definition 2.1 with a Heaviside-type activation function f given in (2) where we assume that $w \in BC^{0,\alpha}(\mathbb{R}^m, L^1(\mathbb{R}^m))$. If an accumulation point u_* of solutions of $u_n - A_n u_n = u_0$ satisfies $\mu(M_{\eta,\rho,\infty}(u_*)) = 0$, then u_* solves the equation $(I - A)u_* = u_0$, i.e. the Volterra integral equation (26) has a solution in X_ρ .

We are now prepared to derive a global existence result with the same technique as in the previous section.

Theorem 2.14 (Global existence for Heaviside-type activation function f)

Consider a kernel w that satisfies the conditions of Definition 2.1 with a Heaviside-type activation function f given in (2) where we assume that $w \in BC^{0,\alpha}(\mathbb{R}^m, L^1(\mathbb{R}^m))$. If an accumulation point u_* of solutions of $u_n - A_n u_n = u_0$ satisfies $\mu(M_{\eta,\infty,\infty}(u_*)) = 0$, then the NFE (4) has a global solution for $t > 0$.

3. Velocity and durability of neural waves

The goal of this part is to estimate the velocity and durability of neural waves [12, 13]. Here, we will say that a wave field *is relevant* at a point $x \in \mathbb{R}^m$ at time $t > 0$ if

$$u(x, t) \geq \eta \tag{66}$$

Otherwise a field is called *irrelevant* in x . The condition (66) arises in connection with the integral Ju given by (9) in (4), where local contributions from $x \in \mathbb{R}^m$ are given only if $u(x, t) \geq \eta$. We will consider the time in which fields that are zero in some part of the space reach a relevant magnitude or amplitude, respectively.

Speed estimates for a neural wave: To evaluate the maximal speed in space of a neural wave we must first define an appropriate setup for the *wave speed*. In our current model setup (4) with a non-local kernel $w(x, y)$ some field $u(x, t) \geq \eta$ has an *instantaneous* effect in the whole space \mathbb{R}^m , since time delay in the propagation of signals is not included into our simple NFE (cf. e.g. [1, 12, 14–16, 19, 21] for a general approach). However, there is a time factor included implicitly by the time derivative $u'(x, t)$ which is modeling the local change of the potential u .

Consider a wave u_0 that is supported in a convex bounded set $M \subset \mathbb{R}^m$ at $t = 0$. Such initial conditions will be called *admissible*. We define the time $T(x)$ as the infimum of all times $t > 0$ for which $u(x, t) \geq \eta$, i.e. $T(x)$ is the minimal time for which the wave reaches the point $x \in \mathbb{R}^m$. Now, the speed of the wave is given by

$$V(x) := \frac{d(x, M)}{T(x)}, \quad x \in \mathbb{R}^m \tag{67}$$

The maximal speed of waves for the NFE (4) is given by

$$V_{\max} := \sup_{u_0 \text{ admissible}, x \in \mathbb{R}^m} |V(x)| \tag{68}$$

A general estimate for the time $T(x)$ is given as follows.

Lemma 3.1

Under the conditions of Definition 2.1, the time $T(x)$ is bounded by

$$T(x) \leq -\tau \log \left(1 - \frac{\eta}{C_w} \right) \tag{69}$$

for all $x \in \mathbb{R}^m$. Given x , it is possible to construct kernels w such that one has equality in (69).

Proof

The quickest increase of the field $u(x, t)$ at a point $x \in \mathbb{R}^m$ is given by the solution of (A1) with initial condition $a=0$ and parameters $b=1/\tau$ and $c=C_w/\tau$. This leads to the equation

$$\eta = C_w(1 - e^{-T(x)/\tau}) \Leftrightarrow T(x) = -\tau \log\left(1 - \frac{\eta}{C_w}\right)$$

which proves the estimate. Here C_w is the supremum over the integrals of $w(x, \cdot)$. For a given open set M and $x \in \mathbb{R}^m$ it is possible to choose kernels such that this supremum is reached at x with $w(x, \cdot)$ supported in M . This proves the second part and the proof is complete. \square

Remark

The previous lemma shows that the conditions of Definition 2.1 are not sufficient to limit the speed of a neural wave. The speed here can become arbitrarily large for $d(x, M) \rightarrow \infty$. However, if we demand further decay properties of the kernel w , the speed will be bounded.

Lemma 3.2

Assume that the kernel $w(x, y)$ satisfies the estimate

$$|w(x, y)| \leq \frac{c}{(1 + |x - y|)^{m+s}}, \quad x \neq y \in \mathbb{R}^m \tag{70}$$

with some constant c and $s \geq 1$. Then the maximal speed of the solutions to (4) is bounded by

$$V_{\max} \leq \frac{C_w - \eta}{s\eta} \tag{71}$$

Proof

From (70) we derive

$$\begin{aligned} \left| \int_M w(x, y) dy \right| &\leq \max\left(C_w, c_m c \int_{d(x, M)}^{\infty} (1+r)^{-(1+s)} dr\right) \\ &= \max\left(C_w, \frac{c_m c}{s \cdot (1+d(x, M))^s}\right) \end{aligned} \tag{72}$$

with some constant c_m depending on the dimension m . For the next steps we will directly work with a bound (72).

On $\mathbb{R}^m \setminus M$ the field was zero at $t=0$. The local behavior of the field is bounded from above by

$$u(x, t) = \frac{c_m c}{(1+d(x, M))^s} \cdot (1 - e^{-t/\tau}), \quad x \in \mathbb{R}^m \setminus M, \quad t \geq 0 \tag{73}$$

After some time T the supremum of the field u on $\mathbb{R}^m \setminus M$ will reach the threshold η , i.e. $\eta = c_m c(1 - e^{-T/\tau})$. We note that the derivative of this field at the boundary ∂M can be estimated via

$$\left. \frac{d}{dr} \frac{\eta}{(1+r)^s} \right|_{r=0} = -s\eta \frac{1}{(1+r)^{s+1}} \Big|_{r=0} = -s\eta \tag{74}$$

Let the boundary ∂M be located at $x=0$ and consider only the one-dimensional case. The field $u(x, T+t)$ for $t=0$ has a tangent $g(x) = \eta - s\eta x$ in $x=0$. The curve has time derivative at $x=0$ bounded by $u' = -\eta + C_w$. Now, we can estimate the speed of the arguments x of $u(x, t) = \eta$ defined in (73) by

$$u(x, T) + u'(x, T) \cdot t \leq \eta - s\eta x + (-\eta + C_w)t \stackrel{!}{=} \eta$$

which yields $x/t = (C_w - \eta)/(\eta s)$ and thus (71). This is a local estimate, but the front with $u(x, t) = \eta$ will move along with the local speed and the above case is an upper estimate for any x and t . This completes the proof. \square

Remark

The speed estimate reflects important properties of the NFE. If the threshold η approaches the maximal forcing term C_w , then the speed will be arbitrarily slow since the fields need more and more time to reach the threshold. If the decay exponent s increases, the speed becomes smaller. If the threshold η is small, then the speed will be large. For $\eta \rightarrow 0$ the speed diverges.

Durability of directed waves: We call a synaptic weight kernel w of the NFE *directed* if there is a direction $d_0 \in \mathbb{S}$ such that

$$w(x, y) \leq 0 \quad \text{for all } (x - y) \cdot d_0 \geq 0 \tag{75}$$

Directedness of a kernel means that its influence to increase a field in some part of space is limited to a direction d with $d \cdot d_0 \geq 0$. We use the notation

$$H(\tau) := \{y \in \mathbb{R}^m : y \cdot d \leq \tau\} \tag{76}$$

for special affine half-spaces in \mathbb{R}^m . We assume to work with non-degenerate kernels in the sense of the condition

$$\int_{H(\tau) \setminus H(\tau_0)} |w(x, y)| dy \rightarrow 0, \quad \tau \rightarrow \tau_0 \quad (77)$$

for $\tau \geq \tau_0$ uniformly for $\tau_0 \in \mathbb{R}$ and $x \in \mathbb{R}^m$, which means that if we sum up all maximal influences over a small strip of depth $s = \tau - \tau_0$ the integral will be small if s is small.

Fields with compactly supported initial conditions that solve the NFE with directed kernel will have a limited durability in any region of space.

Theorem 3.3

Let the conditions of Definition 2.1 be satisfied, the initial field u_0 has compact support in \mathbb{R}^m and let w be a non-degenerated directed synaptic weight kernel with direction $d_0 \in \mathbb{S}$ and the conditions of Definition 2.1. Then, for every $x \in \mathbb{R}^m$ there is a time $T(x) > 0$ such that for $t > T(x)$ the field $u(x, t)$ shows exponential decay.

Proof

Since u_0 is compactly supported there is a parameter τ_0 such that u_0 is zero on $\{x \in \mathbb{R}^m : x \cdot d \leq \tau_0\}$. We choose $\tau_1 > \tau_0$ sufficiently small such that

$$c_1 := \sup_{x \in H(\tau_1)} \int_{H(\tau_1) \setminus H(\tau_0)} |w(x, y)| dy < \eta \quad (78)$$

Then the derivative $u'(x, t)$ for $x \in H(\tau)$ is smaller than $-u(x, t)/\tau + c_1/\tau$ and larger than $-u(x, t)/\tau - c_\tau/\tau$. This means that the function $u(x, t)$ is bounded from above by solutions to Equation (A1) with $b = 1/\tau$ and $c = c_\tau/\tau$, i.e. by

$$u(x, t) = c_\tau + (u_0(x) - c_\tau)e^{-t/\tau}, \quad x \in H(\tau_1), \quad t \geq 0$$

Since $c_1 < \eta$ there is a finite time T depending only on c_1 and η such that $u(y, t) < \eta$ for $t \geq T$ for all $y \in H(\tau_1)$. Then, $f(u(y, t)) = 0$ for $t \geq T$ and $y \in H(\tau_1)$. This means that for $t \geq T$ and $x \in H(\tau_1)$ the field $u(x, t)$ satisfies $u'(x, t) = -u(x, t)/\tau$, which yields exponential convergence toward zero.

We can now repeat the above arguments with τ_2, τ_1 instead of τ_1, τ_0 , where $\tau_2 - \tau_1 = \tau_1 - \tau_0$. Since the field on $H(\tau_0)$ is smaller than η , it will not influence the field in $H(\tau) \setminus H(\tau_0)$ for any $\tau > \tau_0$. This yields some time T_2 such that $u(x, t)$ shows exponential decay in $H(\tau_2)$. Given $x \in \mathbb{R}^m$, after a finite number of applications of the above argument we obtain some time T such that $u(x, t)$ exhibits exponential decay for $t \geq T$. This completes the proof. \square

4. Discussion

In this work we address the solvability of a fundamental NFE, the Amari equation [9], under general assumptions upon the synaptic kernel and activation functions. Using different tools we significantly extend the classical results of Ermentrout and McLeod [13] for the existence of traveling wave solutions, from positive, homogeneous and isotropic kernels $w(x, y)$, which only describe excitatory synapses with translation-invariant weights to the general case of inhomogeneous neural kernel functions. We treat the case of Lipschitz continuous activation functions and provide a detailed discussion of the case of a jump function.

Moreover, our approach complements existent accounts for inhomogeneous kernels with particular symmetries [11, 23, 24] that can be represented by product spaces of the Euclidian space of the nervous tissue with 'feature spaces' such as ocular orientation preference columns. This would be of particular importance for applications in cognitive modeling [27].

However, proving the solvability of NFEs for general kernels is not sufficient in this respect. Algorithms for training or constructing suitable kernel functions have to be devised [28], i.e. the inverse problem for cognitive neurodynamics needs to be solved. We are investigating this issue in the current work.

Appendix A

A.1 Solution to some special ODEs

Here we will briefly summarize results for some special ODEs, which are useful for studying NFEs. First, consider the equation

$$u' = -bu + c \quad (A1)$$

with some positive constant b and $c \in \mathbb{R}$ and the initial condition

$$u(0) = a \in \mathbb{R} \quad (A2)$$

where we assume that $a < c/b$.

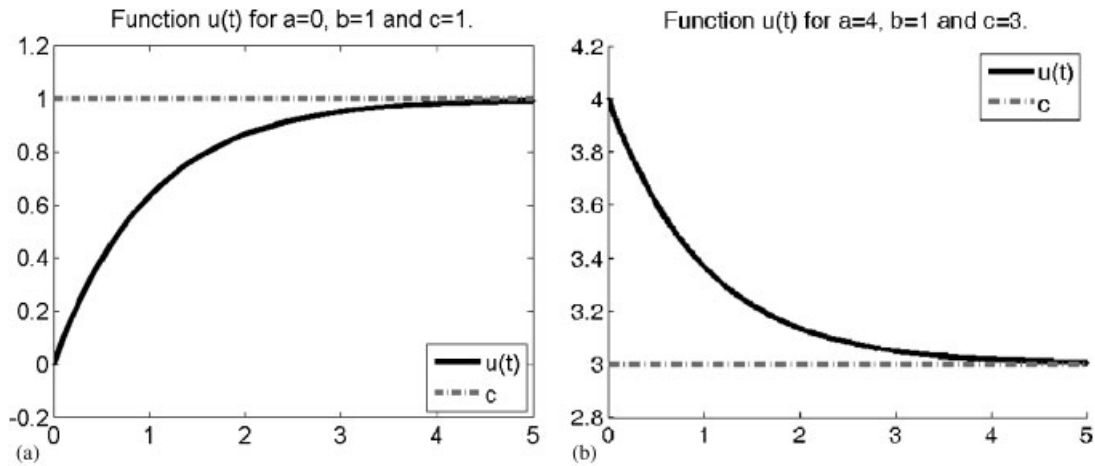


Figure A1. Solutions of the special ordinary differential equation (A1) with two different choices of parameters: (a) $(a, b, c) = (0, 1, 1)$ and (b) $(a, b, c) = (0.7, 1, 3)$. The solution is bounded by c and is exponentially approaching the limiting value $u(t) = c$.

Uniqueness of solutions: First, we investigate uniqueness of the equation. Let u_1, u_2 be solutions and define $u = u_1 - u_2$. Then u solves the homogeneous equation $u' = -bu$ with $u(0) = 0$. Assume that there is some $t > 0$ such that $u(t) \neq 0$. Then we find $\rho \geq 0$ such that $u(t) = 0$ for $t \in [0, \rho]$ and $u(t) \neq 0$ for $t \in (\rho, \sigma)$. Then, we divide by $u(t)$ to obtain

$$\begin{aligned} \frac{u'(t)}{u(t)} &= -b \\ \Rightarrow \log(u(t)) &= -bt + d \\ \Rightarrow u(t) &= e^{-bt+d} \end{aligned} \tag{A3}$$

for $t \in (\rho, \sigma)$ with some integration constant d . However, we need to satisfy the boundary condition $u(\rho) = 0$, which contradicts the positivity of the exponential function e^{-bt+d} for $t, d \in \mathbb{R}$. Thus, the assumption $u(t) \neq 0$ for some $t > 0$ cannot be valid.

Existence of solutions: Solutions can be constructed with a derivation similar to the integration (A3). Under the condition $u(t) \neq c/b$ and $c > 0$ we derive

$$\begin{aligned} u'(t) &= -bu(t) + c = \underbrace{b(c/b - u(t))}_{\geq 0} \\ \Rightarrow \frac{u'(t)}{c/b - u(t)} &= -b \\ \Rightarrow \log(c/b - u(t)) &= -bt + d \\ \Rightarrow u(t) &= c/b - e^{-bt+d} \end{aligned} \tag{A4}$$

Matching the boundary condition $u(0) = a$ for $a < c/b$ yields

$$a = c/b - e^d \Rightarrow e^d = c/b - a \tag{A5}$$

and for $a > c/b$ via $e^{i\pi} = -1$ we obtain a complex $d = i\pi + d_0$ with $e^{d_0} = a - c/b$. Thus, the unique solution is given by

$$u(t) = c/b(1 - e^{-bt}) + ae^{-bt}, \quad t > 0 \tag{A6}$$

The function is shown in Figure A1. For $a = c/b$ the unique solution is given by $u(t) = c/b$. For $c < 0$ it is quickly verified that (A6) satisfies (A1), thus we have unique solvability for $c \in \mathbb{R}$.

Lemma 4.1

The unique solution (A6) to the ODE (A1) for initial value a is bounded by $C_{\text{tot}} := \max(|a|, |c/b|)$.

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